Intoduction to Supergeometry

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Vector superspace

$$V = V_{\bar{0}} \oplus V_{\bar{1}}, \qquad \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

Homogeneous elements: $x \in V_{\bar{0}} \cup V_{\bar{1}}$

$x \in V_{\bar{0}}$ is called even, $|x| = \bar{0}$;

$x \in V_{\bar{1}} \backslash \{0\}$ is called odd, $|x| = \bar{1}$;

The vectors $e_1, \dots, e_{n+m}$ form a basis of V if $e_1, \dots, e_n$ is a basis of $V_{\bar{0}}$ and $e_{n+1}, \dots, e_{n+m}$ is a basis of $V_{\bar{1}}$

$$\dim V = \dim V_{\bar{0}} | \dim V_{\bar{1}} = n|m$$

V and W are vector superspaces $\Rightarrow V \otimes W$ is a vector superspace:

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}})$$

$$(V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}})$$

V and W are vector superspaces $\Rightarrow \mathrm{Hom}(V, W)$ is a vector superspace:

$$\begin{aligned}
\mathrm{Hom}(V, W)_{\bar{0}} &= \mathrm{Hom}(V_{\bar{0}}, W_{\bar{0}}) \oplus \mathrm{Hom}(V_{\bar{1}}, W_{\bar{1}}) \\
&= \{f \in \mathrm{Hom}(V, W) \big| \quad |f(x)| = |x|\} \qquad \text{(morphisms)} \\
\mathrm{Hom}(V, W)_{\bar{1}} &= \mathrm{Hom}(V_{\bar{0}}, W_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, W_{\bar{0}}) \\
&= \{f \in \mathrm{Hom}(V, W) \big| \quad |f(x)| = |x| + \bar{1}, \ x \neq 0\}
\end{aligned}$$

For $V = V_{\bar{0}} \oplus V_{\bar{1}}$ consider the superspace

$$\Pi V = (\Pi V)_{\bar{0}} \oplus (\Pi V)_{\bar{1}} = V_{\bar{1}} \oplus V_{\bar{0}}$$

$\Pi(\Pi V) = V$

A vector supersubspace $W \subset V$ is a vector subspace that is a vector superspace such that

$$W = W_{\bar{0}} \oplus W_{\bar{1}}$$

and

$$W_{\bar{0}} \subset V_{\bar{0}}, \quad W_{\bar{1}} \subset V_{\bar{1}}.$$

Superalgebra

$$A = A_{\bar{0}} \oplus A_{\bar{1}}$$

$$\cdot : A \otimes A \to A$$

$$\cdot \in \mathrm{Hom}_{\bar{0}}(A \otimes A, A), \quad \text{i.e. } |x \cdot y| = |x| + |y|.$$

In other words,

$$A_{\bar{0}} \cdot A_{\bar{0}},\, A_{\bar{1}} \cdot A_{\bar{1}} \subset A_{\bar{0}}, \qquad A_{\bar{0}} \cdot A_{\bar{1}},\, A_{\bar{1}} \cdot A_{\bar{0}} \subset A_{\bar{1}}.$$

A superalgebra A is called *commutative* if

$$xy = (-1)^{|x||y|}yx,$$

$x, y \in V_{\bar{0}} \cup V_{\bar{1}}$.

Sign rule:

If in a formula something of a parity p moves through something of a parity q, then the sign $(-1)^{pq}$ appears.

Example. Commutative algebra: $xy = yx$;

commutative superalgebra: $xy = (-1)^{|x||y|}yx$.

Important example. The Grassmann superalgebra $\Lambda(m)$:

Consider the algebra $\Lambda(m)$ with the generators $1, \xi_1, ..., \xi_m$ and the relations

$$\xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0$$

In particular, $\xi_\alpha^2 = 0$. Any $f \in \Lambda(m)$ has the form

$$f = f_0 + \sum_{r=1}^{m} \sum_{1 \leq \alpha_1 < \cdots < \alpha_r \leq m} f_{\alpha_1 \cdots \alpha_r} \xi_{\alpha_1} \cdots \xi_{\alpha_r}, \quad f_0, f_{\alpha_1 \cdots \alpha_r} \in \mathbb{R}.$$

Let $|1| = \bar{0}$, $|\xi_\alpha| = \bar{1}$ and assume $|xy| = |x| + |y|$. Then $\Lambda(m)$ becomes a commutative superalgebra.

We may start with the vector space $\mathbb{R}^m$ with a basis $\xi_1, \dots, \xi_m$, than the exterior algebra

$$\Lambda(m) = \oplus_{i=0}^{m} \Lambda^i \mathbb{R}^m$$

together with the $\mathbb{Z}_2$-grading

$$\Lambda(m) = \Lambda^{even} \oplus \Lambda^{odd}$$

is the Grassmann superalgebra.

Lie superalgebra:

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}},$$

$$[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \quad |[x,y]| = |x| + |y|$$

1) $[x,y] = -(-1)^{|x||y|}[y,x]$

2) $[[x,y],z] + (-1)^{|x|(|y|+|z|)}[[y,z],x] + (-1)^{|z|(|x|+|y|)}[[z,x],y] = 0$

$\Rightarrow$ $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module

More about the sign rule:

Consider auxiliary anticommuting odd parameters $\eta_1, ..., \eta_N$.

If x_1, x_2... are odd elements, replace them by $\eta_1 x_1$, $\eta_2 x_2$..., and do all computations as usually with even elements. No need to remember the sign rule! Note that then we work not over $\mathbb{R}$, but over $\Lambda(N)$.

Example: get the definition of a commutative superalgebra:

$x, y \in A_{\bar{0}}$, $u, v \in A_{\bar{1}}$. Consider $x, y, \eta_1 u, \eta_2 v$.

$$x(\eta_1 u) = (\eta_1 u)x \Rightarrow \eta_1 xu = \eta_1 ux \Rightarrow xu = ux$$

$$(\eta_1 u)(\eta_2 v) = (\eta_2 v)(\eta_1 u) \Rightarrow \eta_1\eta_2 uv = \eta_2\eta_1 vu$$

$$\Rightarrow \eta_1\eta_2 uv = -\eta_1\eta_2 vu \Rightarrow uv = -vu$$

Recall that $zw = (-1)^{|z||w|} wz, \quad z, w \in A.$

Similarly for a Lie superalgebra $\mathfrak{g}$, let $u, v \in \mathfrak{g}_{\bar{1}}$, then

$$[\eta_1 u, \eta_2 v] = -[\eta_2 v, \eta_1 u] \Rightarrow \eta_1\eta_2[u, v] = -\eta_2\eta_1[v, u]$$

$$\Rightarrow \eta_1\eta_2[u, v] = \eta_1\eta_2[v, u] \Rightarrow [u, v] = [v, u].$$

Example. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\qquad \mathbb{K}^{n|m} = \mathbb{K}^n \oplus \Pi(\mathbb{K}^m)$

$$\mathfrak{gl}(n|m, \mathbb{K}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$$

$$\mathfrak{gl}(n|m, \mathbb{K})_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \simeq \mathfrak{gl}(n, \mathbb{K}) \oplus \mathfrak{gl}(m, \mathbb{K})$$

$$\mathfrak{gl}(n|m)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\} \simeq (\mathbb{K}^n \otimes (\mathbb{K}^m)^*) \oplus ((\mathbb{K}^n)^* \otimes \mathbb{K}^m)$$

$$[X, Y] = XY - (-1)^{|X||Y|} YX$$

Example. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(n|m,\mathbb{R})$.

Define *the supertrace* $\mathrm{str} M = \mathrm{tr} A - \mathrm{tr} D$.

$$\mathfrak{sl}(n|m,\mathbb{R}) = \{M \in \mathfrak{gl}(n|m,\mathbb{R}) | \mathrm{str} M = 0\}.$$

If $m \neq n$, then $\mathfrak{sl}(n|m,\mathbb{R})$ is simple.

For $m = n$ the Lie superalgebra $\mathfrak{psl}(n|n,\mathbb{R}) = \mathfrak{sl}(n|n,\mathbb{R})/\mathbb{R}E_{2n}$ is simple (but it is not a supersubalgebra of $\mathfrak{gl}(n|m,\mathbb{R})$).

Example from differential geometry. Let M be a smooth manifold, X a fixed vector field on M and $\Omega^*(M) = \oplus_{k=0}^n \Omega^k(M)$ the space of differential forms on M. The $\mathbb{R}$-linear operators

$$d,\, L_X,\, i_X \,:\Omega^*(M) \to \Omega^*(M)$$

satisfy

$$L_X = i_X \circ d + d \circ i_X, \quad L_X \circ i_X = i_X \circ L_X, \quad L_X \circ d = d \circ L_X.$$

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, $\mathfrak{g}_{\bar{0}} = \mathbb{R}L_X$, $\mathfrak{g}_{\bar{1}} = \mathbb{R}d \oplus \mathbb{R}i_X$.

Then $\mathfrak{g}$ is a Lie superalgebra with the only non-zero Lie superbracket

$$[d, i_X] = L_X.$$

Bilinear forms on a vector superspace.

Let $g : V \otimes V \to \mathbb{R}$ be a bilinear form on the superspace V.

g is symmetric if $g(y,x) = (-1)^{|x||y|} g(x,y)$;

g is skew-symmetric if $g(y,x) = -(-1)^{|x||y|} g(x,y)$;

g is even if $g(V_{\bar{0}}, V_{\bar{1}}) = g(V_{\bar{1}}, V_{\bar{0}}) = 0$;

g is odd if $g(V_{\bar{0}}, V_{\bar{0}}) = g(V_{\bar{1}}, V_{\bar{1}}) = 0$.

Let g be an even non-degenerate symmetric on

$\mathbb{R}^{n|m} = \mathbb{R}^n \oplus \Pi(\mathbb{R}^m)$,

i.e. $g(\mathbb{R}^n, \Pi(\mathbb{R}^{2k})) = g(\Pi(\mathbb{R}^{2k}), \mathbb{R}^n) = 0$,

the restriction of g to $\mathbb{R}^n$ is non-degenerate and symmetric (with some signature (p, q), $p + q = n$),

the restriction of g to $\Pi(\mathbb{R}^m)$ is non-degenerate and skew-symmetric, i.e. $m = 2k$.

The orthosymplectic Lie superalgebra

$$\mathfrak{osp}(p, q|2k)_{\bar{i}} = \{\xi \in \mathfrak{gl}(n|2k, \mathbb{R})_{\bar{i}} |\, g(\xi x, y) + (-1)^{|x|\bar{i}} g(x, \xi y) = 0\}.$$

Let e.g. the restriction of g to $\mathbb{R}^n$ be positive definite

$$g = \begin{pmatrix} 1_n & 0 & 0 \\ 0 & 0 & 1_k \\ 0 & -1_k & 0 \end{pmatrix}.$$

Then,

$$\mathfrak{osp}(n|2k,\mathbb{R}) = \left\{ \begin{pmatrix} A & B_1 & B_2 \\ B_2^t & C_1 & C_2 \\ -B_1^t & C_3 & -C_1^t \end{pmatrix} \middle| A^t = -A, C_2^t = C_2, C_3^t = C_3 \right\}.$$

$$\mathfrak{osp}(p,q|2k) = (\mathfrak{so}(p,q) \oplus \mathfrak{sp}(2k,\mathbb{R})) \oplus \mathbb{R}^{p,q} \otimes \mathbb{R}^{2k}$$

Consider an odd non-degenerate supersymmetric form g on $\mathbb{R}^{n|n} = \mathbb{R}^n \oplus \Pi(\mathbb{R}^n)$, i.e. $g(\mathbb{R}^n, \mathbb{R}^n) = g(\Pi(\mathbb{R}^n), \Pi(\mathbb{R}^n)) = 0$, and $g(x_0, x_1) = g(x_1, x_0)$ for all $x_0 \in \mathbb{R}^n$, $x_1 \in \Pi(\mathbb{R}^n)$.

There exists a basis of $\mathbb{R}^n \oplus \Pi(\mathbb{R}^n)$ such that $g = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$.

The periplectic Lie superalgebra:

$$\mathfrak{pe}(n, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \middle| B = -B^t, C = C^t \right\}$$

$$\mathfrak{pe}(n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{R}) \oplus (S^2\mathbb{R}^n \oplus \Lambda^2(\mathbb{R}^n)^*)$$

$\mathfrak{spe}(n, \mathbb{R}) = \mathfrak{pe}(n, \mathbb{R}) \cap \mathfrak{sl}(n|n, \mathbb{R})$ is simple.

Consider an odd non-degenerate skew-symmetric form g on $\mathbb{R}^n \oplus \Pi(\mathbb{R}^n)$. There exists a basis of $\mathbb{R}^n \oplus \Pi(\mathbb{R}^n)$ such that

$$g = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

$$\mathfrak{pe}^{sk}(n,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \middle| B = B^t, C = -C^t \right\}.$$

$$\mathfrak{pe}^{sk}(n,\mathbb{R}) \simeq \mathfrak{pe}(n,\mathbb{R}).$$

Let J be an odd complex structure on $\mathbb{R}^{n|n} = \mathbb{R}^n \oplus \Pi(\mathbb{R}^n)$, i.e. J is an odd isomorphism of $\mathbb{R}^n \oplus \Pi(\mathbb{R}^n)$ with $J^2 = -\mathrm{id}$.

The queer Lie superalgebra $\mathfrak{q}(n, \mathbb{R})$ is the subalgebra of $\mathfrak{gl}(n|n, \mathbb{R})$ commuting with J.

There exists a basis of $\mathbb{R}^n \oplus \Pi(\mathbb{R}^n)$ such that $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$.

Then,

$$\mathfrak{q}(n, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}, \quad \mathfrak{sq}(n, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| \mathrm{trB} = 0 \right\}$$

$\mathfrak{psq}(n, \mathbb{R}) = \mathfrak{sq}(n, \mathbb{R})/\mathbb{R}E_{2n}$ is simple.

Examples of exceptional simple Lie superalgebras:

$$\mathfrak{g} = G(3), \quad \mathfrak{g}_{\bar{0}} = G(2) \oplus \mathfrak{sl}(2, \mathbb{C}), \quad \mathfrak{g}_{\bar{1}} = \mathbb{C}^7 \otimes \mathbb{C}^2;$$

$$\mathfrak{g} = F(4), \quad \mathfrak{g}_{\bar{0}} = \mathfrak{spin}(7) \oplus \mathfrak{sl}(2, \mathbb{C}), \quad \mathfrak{g}_{\bar{1}} = \mathbb{C}^8 \otimes \mathbb{C}^2.$$

Let V be a purely odd vector space, i.e. $V = V_{\bar{1}}$. By definition,

$$S^2 V^* = \{b : V \otimes V \to \mathbb{R} \,|\, b(x, y) = (-1)^{|x||y|} b(y, x)\},$$

but $|x| = |y| = \bar{1}$, if $x, y \neq 0$. This shows that $b(x, y) = -b(y, x)$,

$$S^2 V^* = \Lambda^2 \Pi V^*, \qquad S^2 V = \Lambda^2 \Pi V.$$

Similarly,

$$\Lambda^2 V = S^2 \Pi V.$$

The odd vector superspace $\mathbb{R}^{0|m}$ as the first example of a supermanifold 0 Consider $\mathbb{R}^n$. This is both a vector space and a smooth manifolds. The algebra of smooth functions on $\mathbb{R}^n$ contains the dense subset of polynomial functions:

$$S^*(\mathbb{R}^n)^* = \oplus_{k=0}^{\infty} S^k(\mathbb{R}^n)^* \subset C^\infty(\mathbb{R}^n).$$

Consider the odd vector space $\mathbb{R}^{0|m} = \Pi\mathbb{R}^m$. Then

$$S^*(\Pi\mathbb{R}^m)^* = \oplus_{k=0}^{\infty} S^k(\Pi\mathbb{R}^m)^* = \oplus_{k=0}^{\infty} \Lambda^k(\mathbb{R}^m)^* = \Lambda^*(\mathbb{R}^m)^* = \Lambda(m).$$

By this reason,

$$C^\infty(\mathbb{R}^{0|m}) = \Lambda(m).$$

Any $f \in C^\infty(\mathbb{R}^{0|m})$ has the form

$$f = f_0 + \sum_{r=1}^{m} \sum_{1\leq\alpha_1<\cdots<\alpha_r\leq m} f_{\alpha_1\cdots\alpha_r}\xi^{\alpha_1}\cdots\xi^{\alpha_r}, \quad f_0, f_{\alpha_1\cdots\alpha_r} \in \mathbb{R}.$$

The functions ξ^α should play the role of coordinate functions on the "manifold" $\mathbb{R}^{0|m}$. But

$$\xi^\alpha\xi^\beta + \xi^\beta\xi^\alpha = 0, \quad (\xi^\alpha)^2 = 0,$$

i.e. these coordinate functions can not take real values (except 0). Since the coordinate functions should parametrise the points, we get only one point 0 in our "manifold".

By definition, $\mathbb{R}^{0|m}$ is a *supermanifold* of superdimension $0|m$; it is a pair

$$\mathbb{R}^{0|m} = (\{0\}, \Lambda(m)),$$

where 0 is the only point of $\mathbb{R}^{0|m}$ and $\Lambda(m)$ is the algebra of superfunctions on $\mathbb{R}^{0|m}$.

Define the value at the point 0 of the superfunction $f \in C^\infty(\mathbb{R}^{0|m})$ of the form

$$f = f_0 + \sum_{r=1}^{m} \sum_{1 \le \alpha_1 < \cdots < \alpha_r \le m} f_{\alpha_1 \cdots \alpha_r} \xi^{\alpha_1} \cdots \xi^{\alpha_r}, \quad f_0, f_{\alpha_1 \cdots \alpha_r} \in \mathbb{R}$$

by

$$f(0) := f_0 \in \mathbb{R}.$$

Consider the tangent space

$$T_0\mathbb{R}^{0|m} = \{A : C^\infty(\mathbb{R}^{0|m}) \to \mathbb{R} \mid A(fg) = (Af)g(0)+(-1)^{|A||f|}f(0)(Ag)\}.$$

Exercise. The odd vectors $(\partial_\alpha)_0$ acting by $(\partial_\alpha)_0 f = (\partial_\alpha f)_0$ form a basis of $T_0\mathbb{R}^{0|m}$, i.e.

$$T_0\mathbb{R}^{0|m} = \mathbb{R}^{0|m}.$$

Vector fields on $\mathbb{R}^{0|m}$:

$$\mathcal{T}_{\mathbb{R}^{0|m}} = \{A : C^\infty(\mathbb{R}^{0|m}) \to C^\infty(\mathbb{R}^{0|m}) \,|\, A(fg) = (Af)g + (-1)^{|A||f|} f(Ag)\}.$$

Define the odd vectorfields $\frac{\partial}{\partial \xi^\alpha} = \partial_\alpha$ assuming $\partial_\alpha \xi^\beta = \delta_\alpha^\beta$.

Exercise. $\mathcal{T}_{\mathbb{R}^{0|m}} = \Lambda(m) \otimes_{\mathbb{R}} \mathrm{span}_{\mathbb{R}}\{\partial_1, ..., \partial_m\} = \Lambda(m) \otimes_{\mathbb{R}} \mathbb{R}^{0|m}$.

Define the Lie superbrackets by

$$[A, B] = A \circ B - (-1)^{|A||B|} B \circ A.$$

The Lie superalgebra $\mathcal{T}_{\mathbb{R}^{0|m}}$ with this brackets is denoted by $\mathfrak{vect}(0|m, \mathbb{R})$. It is a finite-dimensional Lie superalgebra. For $m \geq 2$ it is **simple**.

For $X = X^\alpha \partial_\alpha \in \mathfrak{vect}(0|m, \mathbb{R})$ define its divergence

$$\mathrm{div} X = \sum_\alpha (-1)^{|X^\alpha|} \partial_\alpha X^\alpha.$$

Define the special (divergence-free) vectorial Lie superalgebra

$$\mathfrak{svect}(0|m) = \{X \in \mathfrak{vect}(0|m, \mathbb{R}) \mid \mathrm{div} X = 0\}.$$

It is simple for $m \geq 3$.

Let $m = 2k$. Consider the 2-form $\omega = \sum_{\alpha=1}^{k} d\xi^\alpha \circ d\xi^{\alpha+k}$. Assume $|d\xi^\alpha| = \bar{0}$, $d\xi^\alpha \circ d\xi^\beta = d\xi^\beta d\xi^\alpha$.

Define the Lie superalgebra of Hamiltonian vector fields

$$\tilde{\mathfrak{h}}(0|2k, \mathbb{R}) = \{X \in \mathfrak{vect}(0|2k, \mathbb{R}) \mid L_X\omega = 0\}.$$

The Lie superalgebra $\mathfrak{h}(0|2k, \mathbb{R}) = [\tilde{\mathfrak{h}}(0|2k, \mathbb{R}), \tilde{\mathfrak{h}}(0|2k, \mathbb{R})]$ is simple.

Classification of finite dim. simple complex Lie superalgebras:

• classical type, i.e. the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is completely reducible

$\mathfrak{sl}(n|m,\mathbb{C})$, $\mathfrak{psl}(n|n,\mathbb{C})$, $\mathfrak{osp}(n|2m,\mathbb{C})$, $\mathfrak{pe}(n,\mathbb{C})$, $G(3)$, $F(4)$,...

• Cartan type

$\mathfrak{vect}(0|n,\mathbb{C})$, $\mathfrak{svect}(0|n,\mathbb{C})$, $\mathfrak{h}(0|2k,\mathbb{C})$...

V. G. Kac, *Lie superalgebras.* Adv. Math., 26 (1977), 8–96.

L. Frappat, A. Sciarrino, P. Sorba, *Dictionary on Lie Superalgebras*, arXiv:hep-th/9607161

Peculiarities:

- zero Killing form e.g. on $\mathfrak{psl}(n|n, \mathbb{C})$, $\mathfrak{pe}(n, \mathbb{C})$;
- in general no total reducibility of simple LSA;
- semisimple LSA are of the from $\sum \mathfrak{g}_i \otimes \Lambda(n_i)$;
- there exist non-trivial irreducible representation of solvable LSA

The state of a quantum mechanical system is represented by a unit vector (defined up to a phase, i.e. a complex number of length 1) in a complex Hilbert space H.

Let H describe the state of a single particle. Then the states of two identical particles v and v' is described by the tensor product

$$H \otimes H.$$

Since the particles are identical, the states

$$v \otimes v' \quad \text{and} \quad v' \otimes v$$

must be the same.

But the state is defined up to a phase, consequently

$$v' \otimes v = \lambda v \otimes v'.$$

Applying this twice, we get $\lambda^2 = 1$, i.e. $\lambda = \pm 1$.

If $\lambda = 1$, then the particle is called *boson*. Two identical bosons are described by a vector in $S^2 H$.

If $\lambda = -1$, then the particle is called *fermion*. Two identical fermions are described by a vector in $\Lambda^2 H$.

To unify the bosons and fermions consider the Hilbert superspace

$$H = H_{\bar{0}} \oplus H_{\bar{1}},$$

Where $H_{\bar{0}}$ describes a boson and $\Pi H_{\bar{1}}$ describes a fermion.

Then

$$S^2 H = S^2 H_{\bar{0}} \bigoplus H_{\bar{0}} \otimes H_{\bar{1}} \bigoplus S^2 H_{\bar{1}}.$$

But $S^2 H_{\bar{1}} = \Lambda^2 \Pi H_{\bar{1}}$.

Thus the summands of $S^2 H$ describe two bosons, or a boson and a fermion, or two fermions.

The sign rule of superalgebra encodes the statistics of a particle!

Let A be a supercommutative superalgebra and M be a real vector superspace.

M is a left A-supermodule if there exists a morphism

$$\cdot : A \otimes_{\mathbb{R}} M \to M, \quad (a, x) \mapsto a \cdot x, \quad |a \cdot x| = |a| + |x|.$$

M can be also considered as a right A-supermodule if we put

$$x \cdot a = (-1)^{|x||a|} a \cdot x.$$

Let M and N be A-supermodules.

A homogeneous map $\varphi : M \to N$ is called A-linear if

$$\varphi(ax) = (-1)^{|\varphi||a|}a\varphi(x).$$

Equivalently,

$$\varphi(xa) = \varphi(x)a.$$

Denote by $\mathrm{Hom}_A(M, N)$ the vector superspace of all A-linear maps from M to N, and set $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$.

We say that M over A is free of rank $n|m$ if there exists a basis $e_1, \dots, e_{n+m}$ of M over A such that $e_1, \dots, e_n \in M_{\bar{0}}$ and $e_{n+1}, \dots, e_{n+m} \in M_{\bar{1}}$.

This means that for any $x \in M$ there exist $x^1, \dots, x^{n+m} \in A$ such that

$$x = \sum_{a=1}^{n+m} x^a e_a.$$

Let M and N be free A-supermodules of ranks $m|n$ and $r|s$. For an A-linear map $\varphi : M \to N$ define $\varphi_a^b \in A$, $a = 1, ..., n+m$, $b = 1, ..., r+s$ such that

$$\varphi(e_a) = \sum_{b=1}^{r+s} f_b \varphi_a^b.$$

We get an $r+s \times n+m$ matrix with elements from A.

Let $x = \sum_{a=1}^{n+m} e_a x^a \in M$, $y = \varphi(x) = \sum_{b=1}^{r+s} f_b y^b \in N$ then

$$\varphi(x) = \varphi\left(\sum_{a=1}^{n+m} e_a x^a\right) = \sum_{a=1}^{n+m} \varphi(e_a) x^a = \sum_{a=1}^{n+m} \sum_{b=1}^{r+s} f_b \varphi_a^b x^a.$$

We get that

$$y^b = \sum_{a=1}^{n+m} \varphi_a^b x^a.$$

In the matrix form

$$\begin{pmatrix} y^1 \\ \vdots \\ y^{r+s} \end{pmatrix} = \begin{pmatrix} \varphi_1^1 & \cdots & \varphi_{n+m}^1 \\ \vdots & & \vdots \\ \varphi_1^{r+s} & \cdots & \varphi_{n+m}^{r+s} \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \vdots \\ x^{n+m} \end{pmatrix}.$$

Since we have the decompositions $M = M_{\bar{0}} \oplus M_{\bar{1}}$ and $N = N_{\bar{0}} \oplus N_{\bar{1}}$, the map φ can be divided into 4 parts. According to that we may write

$$\varphi = \begin{pmatrix} \varphi_{\bar{0}\bar{0}} & \varphi_{\bar{0}\bar{1}} \\ \varphi_{\bar{1}\bar{0}} & \varphi_{\bar{1}\bar{1}} \end{pmatrix} = \begin{pmatrix} \varphi_1^1 & \cdots & \varphi_{n+m}^1 \\ \vdots & & \vdots \\ \varphi_1^{r+s} & \cdots & \varphi_{n+m}^{r+s} \end{pmatrix}.$$

φ is even if and only if the entries of the matrices $\varphi_{\bar{0}\bar{0}}$ and $\varphi_{\bar{1}\bar{1}}$ are even and the entries of the matrices $\varphi_{\bar{1}\bar{0}}$ and $\varphi_{\bar{0}\bar{1}}$ are odd.

The dual space: $M^* = \mathrm{Hom}_A(M, A)$.

For $\varphi \in \mathrm{Hom}_A(M, N)$ define $\varphi^* \in \mathrm{Hom}_A(N^*, M^*)$,

$$\varphi^*(\xi) = (-1)^{|\varphi||\xi|}\xi \circ \varphi.$$

Then the matrix of φ^* w.r.t. the dual bases f_b^* and e_a^* has the form (exercise)

$$\begin{pmatrix} \varphi^t_{\bar{0}\bar{0}} & (-1)^{|\varphi|+1}\varphi^t_{\bar{1}\bar{0}} \\ (-1)^{|\varphi|}\varphi^t_{\bar{0}\bar{1}} & \varphi^t_{\bar{1}\bar{1}} \end{pmatrix}.$$

Let L be an $r+s \times n+m$ matrix with elements form A

$$L = \begin{pmatrix} L_{\bar{0}\bar{0}} & L_{\bar{0}\bar{1}} \\ L_{\bar{1}\bar{0}} & L_{\bar{1}\bar{1}} \end{pmatrix}$$

(i.e. it can be the matrix of a homomorphism from M to N)

We say that L is even if the entries of the matrices $L_{\bar{0}\bar{0}}$ and $L_{\bar{1}\bar{1}}$ are even and the entries of the matrices $L_{\bar{1}\bar{0}}$ and $L_{\bar{0}\bar{1}}$ are odd.

Define the supertransposed matrix

$$L^{st} = \begin{pmatrix} L^t_{\bar{0}\bar{0}} & (-1)^{|L|+1} L^t_{\bar{1}\bar{0}} \\ (-1)^{|L|} L^t_{\bar{0}\bar{1}} & L^t_{\bar{1}\bar{1}} \end{pmatrix}.$$

Consider set $\mathrm{Mat}_A(n|m)$ of all squire matrices of order $n+m$ with elements from A. It becomes an A-supermodule with respect to the multiplication

$$aL = \begin{pmatrix} aL_{\bar{0}\bar{0}} & aL_{\bar{0}\bar{1}} \\ (-1)^{|a|}aL_{\bar{1}\bar{0}} & (-1)^{|a|}aL_{\bar{1}\bar{1}} \end{pmatrix}.$$

For a homogenious $L = \begin{pmatrix} L_{\bar{0}\bar{0}} & L_{\bar{0}\bar{1}} \\ L_{\bar{1}\bar{0}} & L_{\bar{1}\bar{1}} \end{pmatrix}$ define the *supertrace*

$$\mathrm{str} L = \mathrm{tr}\, L_{\bar{0}\bar{0}} - (-1)^{|L|}\, \mathrm{tr}\, L_{\bar{1}\bar{1}}.$$

Proposition. $\mathrm{str}([K, L]) = 0$.

The group

$$\mathrm{GL}_A(n|m) = \{L \in \mathrm{Mat}_A(n|m) \,|\, |L| = \bar{0},\ L \text{ is invertible}\}$$

is called *general linear supergroup* of rank $n|m$ over A.

Example. $\mathrm{GL}_{\mathbb{R}}(n|m) = \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(m, \mathbb{R})$.

Theorem. Let $L \in \mathrm{Mat}_A(n|m)$. Then $L \in \mathrm{GL}_A(n|m)$ if and only if $L_{\bar{0}\bar{0}}$ and $L_{\bar{1}\bar{1}}$ are invertible.

Let $B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}$ be a usual real matrix. Suppose that B_{11} is invertible, then

$$B = \begin{pmatrix} 1 & B_{01}B_{11}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_{00} - B_{01}B_{11}^{-1}B_{10} & 0 \\ B_{10} & B_{11} \end{pmatrix},$$

consequently,

$$\det B = \det(B_{00} - B_{01}B_{11}^{-1}B_{10}) \cdot \det B_{11}.$$

For $L \in \mathrm{GL}_A(n|m)$ define its **superdeterminant** or **Berezian**

$$\mathrm{Ber} \mathrm{L} = \det(L_{\bar{0}\bar{0}} - L_{\bar{0}\bar{1}} L_{\bar{1}\bar{1}}^{-1} L_{\bar{1}\bar{0}}) \cdot \det L_{\bar{1}\bar{1}}^{-1} \in A_{\bar{0}}.$$

Theorem. $\mathrm{Ber}(KL) = \mathrm{Ber}(K) \cdot \mathrm{Ber}(L)$.

$\mathrm{Ber}(E_{n+m} + \epsilon L) = 1 + \epsilon \mathrm{str} L$, $\epsilon^2 = 0$.

$\mathrm{Ber} \exp L = e^{\mathrm{str} L}$.

A superdomain of dimension $n|m$

$$\mathcal{U} = (U, C^\infty(\mathcal{U})), \quad U \subset \mathbb{R}^n, \quad C^\infty(\mathcal{U}) = C^\infty(U) \otimes \Lambda(m).$$

Let $\xi^1, ..., \xi^m$ be generators of $\Lambda(m)$, then any $f \in C^\infty(\mathcal{U})$ can be written as

$$f = \tilde{f} + \sum_{r=1}^{m} \sum_{\alpha_1 < \cdots < \alpha_r} f_{\alpha_1 \ldots \alpha_r} \xi^{\alpha_1} \cdots \xi^{\alpha_r}, \quad \tilde{f}, f_{\alpha_1 \ldots \alpha_r} \in C^\infty(U).$$

$$x \in U \quad \Rightarrow \quad f(x) := \tilde{f}(x) \in \mathbb{R}.$$

A morphism of superdomains:

$$\varphi : \mathcal{U} = (U, C^\infty(\mathcal{U})) \to \mathcal{V} = (V, C^\infty(\mathcal{V}))$$

is a pair

$$\varphi = (\tilde{\varphi}, \varphi^*), \quad \tilde{\varphi} : U \to V, \quad \varphi^* : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$$

such that

$$(\varphi^* f)(x) = f(\tilde{\varphi}(x)).$$

If $\psi = (\tilde{\psi}, \psi^*) : \mathcal{V} \to \mathcal{W}$ is another morphism, then the decomposition is defined as

$$\psi \circ \varphi = (\tilde{\psi} \circ \tilde{\varphi}, \varphi^* \circ \psi^*) : \mathcal{U} \to \mathcal{W}.$$

$\varphi : \mathcal{U} \to \mathcal{V}$ is called a diffeomorphism if it admits an inverse morphism.

Example.

The inclusion

$$i = (\tilde{i}, i^*) : U \to \mathcal{U}, \quad \tilde{i}(x) = x, \quad i^*(f) = \tilde{f}.$$

The projection

$$p = (\tilde{p}, p^*) : \mathcal{U} \to U, \quad \tilde{p}(x) = x, \quad p^*(f) = f.$$

Proposition. For any morphism of superalgebras $\varphi^* : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ there exists a unique continuous map $\tilde{\varphi} : U \to V$ such that $\varphi = (\tilde{\varphi}, \varphi^*)$ is a morphism from $\mathcal{U}$ to $\mathcal{V}$.

Proof. The composition

$$C^\infty(V) \to C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U}) \to C^\infty(U)$$

defines map $\varphi : U \to V$, which is compatible with φ^*.

Corollary. For any morphism $s : C^\infty(\mathcal{U}) \to \mathbb{R}$ there exists a unique point $x \in U$ such that $s(f) = f(x)$.

Proof. Since $\mathbb{R} = C^\infty(\mathrm{pt})$, $\varphi^* = s$ defines $\varphi : \mathrm{pt} \to \mathcal{U}$.

Let $x = \tilde{\varphi}(\mathrm{pt}) \in U$.

Since $\varphi^*(f) \in \mathbb{R}$,

$$\varphi^*(f) = \varphi^*(f)(\mathrm{pt}) = f(\tilde{\varphi}(\mathrm{pt})) = f(x).$$

Systems of coordinates.

Consider a superdomain $\mathcal{U} = (U, C^\infty(\mathcal{U}) = C^\infty(U) \otimes \Lambda(m))$.

Let $x^1, ..., x^n$ be coordinates on U; $\xi^1, ..., \xi^m$ odd generators of $\Lambda(m)$.

The superfunctions $x^1, ..., x^n, \xi^1, ..., \xi^m$ are called coordinates on $\mathcal{U}$.

Denotation (x^i, ξ^α), or (x^a), $\quad x^{n+\alpha} = \xi^\alpha$.

Vector fields on $\mathcal{U}$. $\mathcal{T}_{\mathcal{U}} = (\mathcal{T}_{\mathcal{U}})_{\bar{0}} \oplus (\mathcal{T}_{\mathcal{U}})_{\bar{1}}$,

$$(\mathcal{T}_{\mathcal{U}})_{\bar{i}} = \left\{ X : C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U}) \,\middle|\, \begin{array}{c} |X| = \bar{i},\ X \text{ is } \mathbb{R}\text{-linear} \\ X(fg) = X(f)g + (-1)^{|f||X|} f X(g) \end{array} \right\}$$

Define the vector fields ∂_{x^i} and ∂_{ξ^α} assuming

$$\partial_{x^i}(f\xi^{\alpha_1}\cdots\xi^{\alpha_r}) = \frac{\partial f}{\partial x^i}\xi^{\alpha_1}\cdots\xi^{\alpha_r},$$

$$\partial_{\xi^\alpha}(f\xi^{\alpha_1}\cdots\xi^{\alpha_r}) = \sum_{s=1}^{r}(-1)^{s-1}\delta^{\alpha\alpha_s} f\xi^{\alpha_1}\cdots\widehat{\xi^{\alpha_s}}\cdots\xi^{\alpha_r}.$$

Proposition. The $C^\infty(\mathcal{U})$-module $\mathcal{T}_\mathcal{U}$ is free of rank $n|m$.

$\mathcal{T}_\mathcal{U} = C^\infty(\mathcal{U}) \otimes_\mathbb{R} \mathrm{span}_\mathbb{R}\{\partial_{x^1}, ..., \partial_{\xi^m}\}$.

Proof. Let $X \in \mathcal{T}_\mathcal{U}$. We claim that $X = (Xx^a)\partial_a$.

Consider

$X' = X - (Xx^a)\partial_a, \quad X'(fg) = X'(f)g + (-1)^{|f||X'|}fX'(g)$.

For $f \in C^\infty(U)$ let $X'(f) = \sum X'_{\alpha_1,\dots,\alpha_r}(f)\xi^{\alpha_1}\cdots\xi^{\alpha_r}$,

then $X'_{\alpha_1,\dots,\alpha_r} : C^\infty(U) \to C^\infty(U)$,

$X'_{\alpha_1,\dots,\alpha_r}(fg) = X'_{\alpha_1,\dots,\alpha_r}(f)g + fX'_{\alpha_1,\dots,\alpha_r}(g), \quad X'_{\alpha_1,\dots,\alpha_r}(x^i) = 0$,

$\Longrightarrow \quad X'_{\alpha_1,\dots,\alpha_r} = 0, \quad X'(f) = 0$.

Moreover, $X'(\xi^\alpha) = 0 \quad \Longrightarrow \quad X' = 0$.

Lemma. Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a morphism, then

$$\frac{\partial}{\partial x^a}(\varphi^* f) = \sum_b \frac{\partial \varphi^*(y^b)}{\partial x^a} \varphi^* \left(\frac{\partial f}{\partial y^b} \right),$$

$f \in C^\infty(\mathcal{V})$.

Theorem. If $\varphi : \mathcal{U} \to \mathcal{V}$ is a morphism and $y^1, ..., y^r, \eta^1, ..., \eta^s$ are coordinates on $\mathcal{V}$, then the functions $\varphi^*(y^1), ..., \varphi^*(y^r), \varphi^*(\eta^1), ..., \varphi^*(\eta^s)$ uniquely define φ.

Proof. Note: if $g = \sum g_{\alpha_1,...,\alpha_p} \xi^{\alpha_1} \cdots \xi^{\alpha_p} \in C^\infty(\mathcal{U})$, then $g_{\alpha_1,...,\alpha_p} = (\partial_{\xi^{\alpha_p}} \cdots \partial_{\xi^{\alpha_1}} g)^\sim$.

First let $f = f(y^1, ..., y^r) \in C^\infty(V)$, then we may find $\varphi^*(f)$ using the previous formula and the lemma:

e.g. $\partial_{\xi^\alpha} \varphi^*(f) = \sum_b \frac{\partial \varphi^*(y^b)}{\partial \xi^\alpha} \varphi^* \left(\frac{\partial f}{\partial y^b} \right) = \sum_b \frac{\partial \varphi^*(y^i)}{\partial \xi^\alpha} \varphi^* \left(\frac{\partial f}{\partial y^i} \right)$.

In general, if $f = \sum f_{\beta_1,...,\beta_p} \theta^{\beta_1} \cdots \theta^{\beta_p} \in C^\infty(\mathcal{V})$, then $\varphi^*(f) = \sum \varphi^*(f_{\beta_1,...,\beta_p}) \varphi^*(\theta^{\beta_1}) \cdots \varphi^*(\theta^{\beta_p})$.

This gives the so-called **symbolic way of calculation**: if $\mathcal{U}$ and $\mathcal{V}$ are superdomains with coordinates $(x,\xi)=(x^i,\xi^\alpha)$ and $(y,\theta)=(y^k,\theta^\beta)$, a morphism $\varphi:\mathcal{U}\to\mathcal{V}$ can be written symbolically

$$\varphi:(x,\xi)\mapsto(y,\theta),\quad y=y(x,\xi),\,\theta=\theta(x,\xi),$$

where in fact $y^k=\varphi^*(y^k)=y^k(x^i,\xi^\alpha)$, $\theta^\beta=\varphi^*(\theta^\beta)=\theta^\beta(x^i,\xi^\alpha)$.

We may write $\varphi^*(f)(x^i,\xi^\alpha)=f(y^j(x^i,\xi^\alpha),\theta(x^i,\xi^\alpha))$ and find this function using the above proof.

Example. Let $\mathcal{U}=\mathcal{V}=\mathbb{R}^{1|2}$ with the coordinates x,ξ^1,ξ^2 and φ is given by

$$\varphi^*(x)=x+\xi^1\xi^2,\quad \varphi^*(\xi^1)=\xi^1,\quad \varphi^*(\xi^2)=\xi^2.$$

Let $f=f(x)$, then

$f(x+\xi^1\xi^2)=(\varphi^*f)(x,\xi^1,\xi^2)$,

$(\varphi^*f)(x,\xi^1,\xi^2)=(\varphi^*f)^\sim(x)+(\varphi^*f)_{12}(x)\xi^1\xi^2$,

$(\varphi^*f)^\sim(x)=f(x)$,

$$(\varphi^* f)_{12} = (\partial_{\xi^2}\partial_{\xi^1}\varphi^*(f))^\sim = (\partial_{\xi^2}(\partial_{\xi^1}(\varphi^*(x))\varphi^*(\partial_x f)))^\sim =$$
$$(\partial_{\xi^2}(\xi^2\varphi^*(\partial_x f)))^\sim = (\varphi^*(\partial_x f))^\sim - (\xi^2\partial_{\xi^2}\varphi^*(\partial_x f))^\sim = \partial_x f.$$

Thus, $f(x+\xi^1\xi^2) = f(x) + \partial_x f(x)\xi^1\xi^2$

We see that if $f \in C^\infty(\mathcal{V}^{r|s})$, then we may consider the expression

$$f(g_1, ..., g_r, h_1, ..., h_s),$$

where $g_1, ..., g_r$ and $h_1, ..., h_r$ are respectively even and odd functions on some $\mathcal{U}$.

Let $x^1, ..., x^n, \xi^1, ..., \xi^m$ be coordinates on $\mathcal{U}$. If $\varphi : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism and $y^1, ..., y^n, \eta^1, ..., \eta^m$ are coordinates on $\mathcal{V}$ as above,

then the functions $\varphi^*(y^1), ..., \varphi^*(y^n), \varphi^*(\eta^1), ..., \varphi^*(\eta^m)$ are also called coordinates on $\mathcal{U}$.

In that case $\varphi^*(y^1), ..., \varphi^*(y^n)$ are not necessary coordinates on U.

By the above considerations, the expression

$f(y^j, \theta^\beta) = f(x^i(y^j, \theta^\beta), \xi^\alpha(y^j, \theta^\beta))$ makes sense.

Examples of morphisms.

1. $\varphi : \mathbb{R}^n \to \mathbb{R}^{k|m}$:

since $(\theta^\beta)^2 = 0, \quad (\varphi^*(\theta^\beta))^2 = 0$,

but $\varphi^*(\theta^\beta) \in C^\infty(\mathbb{R}^n) \Longrightarrow \varphi^*(\theta^\beta) = 0$,

thus φ is given by $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^k$.

2. $\varphi : \mathbb{R}^{0|2} \to M^{n|0}$:

$$f \in C^\infty M \Longrightarrow \quad \varphi^*(f) = a(f) + b(f)\xi^1\xi^2, \quad a(f), b(f) \in \mathbb{R}.$$

$$\varphi^*(fg) = \varphi^*(f)\varphi^*(g) \Longrightarrow a(fg) + b(fg)\xi^1\xi^2 =$$

$$a(f)b(f) + (a(g)b(f) + a(f)b(g))\xi^1\xi^2,$$

$$\Longrightarrow a(fg) = a(f)a(g), \quad b(fg) = a(g)b(f) + a(f)b(g)$$

$\Longrightarrow a : C^\infty M \to \mathbb{R}$ is a homomorphism $\Longrightarrow \exists x \in M$, $a(f) = f(x)$,

finally, $b(fg) = b(g)f(x) + f(x)b(g)$, i.e. $b \in T_x M$.

Thus, φ is defined by a point $x \in m$ and a tangent vector $b \in T_x M$, $\varphi^*(f) = f(x) + b(f)\xi^1\xi^2$.

Example. Let $E \to U$ be a vector bundle over U,

$$\mathcal{U} = (U, \Gamma(U, \Lambda E)).$$

If $\xi^1, ..., \xi^m$ are generators of $\Gamma(U, \Lambda E)$, then $x^1, ..., x^n, \xi^1, ..., \xi^m$ are coordinates on $\mathcal{U}$.

Any automorphism φ of the bundle $\Lambda E \to U$ preserving the parity defines the automorphism of $\mathcal{U}$:

$$\varphi^*(x^i) = \varphi^0(x^1, ..., x^n),$$

$$\varphi^*(\xi^\alpha) = \sum_{r \geq 0} \sum_{\alpha_1 < \cdots < \alpha_{2r+1}} \varphi^\alpha_{\alpha_1 \dots \alpha_{2r+1}}(x^1, ..., x^n) \xi^{\alpha_1} \cdots \xi^{\alpha_{2r+1}}.$$

Any morphism of $\mathcal{U}$ has the coordinate form

$$\varphi^*(x^i) = \varphi^0(x^1, ..., x^n) + \sum_{r\geq 1} \sum_{\alpha_1<\cdots<\alpha_{2r}} \varphi^{\alpha}_{\alpha_1\dots\alpha_{2r}}(x^1, ..., x^n)\xi^{\alpha_1}\cdots\xi^{\alpha_{2r}},$$

$$\varphi^*(\xi^\alpha) = \sum_{r\geq 0} \sum_{\alpha_1<\cdots<\alpha_{2r+1}} \varphi^{\alpha}_{\alpha_1\dots\alpha_{2r+1}}(x^1, ..., x^n)\xi^{\alpha_1}\cdots\xi^{\alpha_{2r+1}}.$$

Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a morphism and $X \in \mathcal{T}_{\mathcal{U}}$. We get the map $X \circ \varphi^* : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$.

Lemma. $\left(\frac{\partial}{\partial x^a} \circ \varphi^*\right) f = \sum_b \frac{\partial \varphi^*(y^b)}{\partial x^a} \varphi^* \left(\frac{\partial f}{\partial y^b}\right)$, $f \in C^\infty(\mathcal{V})$.

In the matrix form:

$$\begin{pmatrix} \frac{\partial(\varphi^* f)}{\partial x^i} \\ \frac{\partial(\varphi^* f)}{\partial \xi^\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\partial(\varphi^* y^j)}{\partial x^i} & \frac{\partial(\varphi^* \eta^\beta)}{\partial x^i} \\ \frac{\partial(\varphi^* y^j)}{\partial \xi^\alpha} & \frac{\partial(\varphi^* \eta^\beta)}{\partial \xi^\alpha} \end{pmatrix} \cdot \begin{pmatrix} \varphi^* \frac{\partial f}{\partial y^j} \\ \varphi^* \frac{\partial f}{\partial \eta^\beta} \end{pmatrix}.$$

Define the Jacoby matrix of φ:

$$J(\varphi) = \begin{pmatrix} \frac{\partial(\varphi^* y^j)}{\partial x^i} & \frac{\partial(\varphi^* \eta^\beta)}{\partial x^i} \\ \frac{\partial(\varphi^* y^j)}{\partial \xi^\alpha} & \frac{\partial(\varphi^* \eta^\beta)}{\partial \xi^\alpha} \end{pmatrix}^{st} =$$

$$\begin{pmatrix} \frac{\partial(\varphi^* y^1)}{\partial x^1} & \dots & \frac{\partial(\varphi^* y^1)}{\partial x^n} & -\frac{\partial(\varphi^* y^1)}{\partial \xi^1} & \dots & -\frac{\partial(\varphi^* y^1)}{\partial \xi^m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial(\varphi^* y^r)}{\partial x^1} & \dots & \frac{\partial(\varphi^* y^r)}{\partial x^n} & -\frac{\partial(\varphi^* y^r)}{\partial \xi^1} & \dots & -\frac{\partial(\varphi^* y^r)}{\partial \xi^m} \\ \frac{\partial(\varphi^* \eta^1)}{\partial x^1} & \dots & \frac{\partial(\varphi^* \eta^1)}{\partial x^n} & \frac{\partial(\varphi^* \eta^1)}{\partial \xi^1} & \dots & \frac{\partial(\varphi^* \eta^1)}{\partial \xi^m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial(\varphi^* \eta^s)}{\partial x^1} & \dots & \frac{\partial(\varphi^* \eta^s)}{\partial x^n} & \frac{\partial(\varphi^* \eta^s)}{\partial \xi^1} & \dots & \frac{\partial(\varphi^* \eta^s)}{\partial \xi^m} \end{pmatrix}.$$

Lemma. If $\varphi : \mathcal{U} \to \mathcal{V}$ and $\psi : \mathcal{V} \to \mathcal{W}$ are morphisms, then

$$J(\psi \circ \varphi) = \varphi^*(J(\psi)) \cdot J(\varphi).$$

Berezin integral.

Let $x^1, ..., x^n, \xi^1, ..., \xi^m$ be coordinates on $\mathcal{U}$ such that $x^1, ..., x^n$ are coordinates on U; let $f \in C^\infty(\mathcal{U})$. to define $\int_{\mathcal{U}} f$ assume the following:

$$\int d\xi^\alpha = 0, \quad \int \xi^\alpha d\xi^\alpha = 1, \quad \xi^\alpha d\xi^\beta = -d\xi^\beta \cdot \xi^\alpha, \quad \xi^\alpha dx^i = dx^i \cdot \xi^\alpha.$$

Using that, we get

$$\int_{\mathcal{U}} dx^1 \cdots dx^n d\xi^1 \cdots d\xi^m f = (-1)^{\frac{m(m-1)}{2}} \int_U dx^1 \cdots dx^n f_{1\dots m}.$$

Note that

$$\int_{\mathcal{U}} dx^1 \cdots dx^n d\xi^1 \cdots d\xi^m f = \int_U dx^1 \cdots dx^n \partial_{\xi^1} \cdots \partial_{\xi^m} f.$$

Theorem. Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a diffeomorphism of superdomains.

Let $f \in C^\infty(\mathcal{V})$ have a compact support. Then

$$\int_{\mathcal{V}} f = \int_{\mathcal{U}} \varphi^* f \cdot \mathrm{Ber}(J(\varphi)).$$

Sheaves. Let M be a topological space. A sheaf $\mathcal{F}$ of algebras (vector spaces, groups,...) on M is an assignment

$$U \mapsto \mathcal{F}(U)$$

to each open subset $U \subset M$ of an algebra (vector space, group) $\mathcal{F}(U)$ such that the following conditions are satisfied.

If $V \subset U$, then there exists a homomorphism map

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V), \quad f \mapsto \rho_{U,V}(f)$$

such that **1)** $\rho_{U,U} = \mathrm{id}$; **2)** $\rho_{W,V} = \rho_{U,V} \circ \rho_{W,U}, \quad V \subset U \subset W$

3) if (U_i) is a covering of U, $f_i \in \mathcal{F}(U_i)$, $\rho_{U_i,U_i \cap U_j}(f_i) = \rho_{U_j,U_i \cap U_j}(f_j)$, then there exists a unique $f \in \mathcal{F}(U)$ such that $\rho_{U,U_i} f = f_i$.

A morphism $\varphi : \mathcal{F} \to \mathcal{T}$ of two sheaves on M is a collection of maps

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{T}(U),$$

$U \subset M$ is open such that

$$r_{U,V} \circ \varphi(U) = \varphi(V) \circ \rho_{U,V}, \quad V \subset U.$$

Example. M is a smooth manifold, and C^∞_M is the sheaf of smooth functions on M: $C^\infty_M(U)$ are smooth functions on the subset $U \subset M$.

Note that a smooth manifolds may be defined as a pair (M, C^∞_M), where M is a Hausdorf topological space, and C^∞_M is a sheaf of commutative algebras on M locally isomorphic to the sheaf of smooth functions on an open subset of $\mathbb{R}^n$.

Example. $E \to M$ is a vector bundle over a smooth manifold M,

$U \mapsto \Gamma(U, E)$ is the sheaf of smooth sections of E.

Note that this sheaf allows to reconstruct E.

Definition of a supermanifold:

A supermanifold of dimension $n|m$ is a pair $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, where M is a Hausdorf topological space, and $\mathcal{O}_{\mathcal{M}}$ is a sheaf of commutative superalgebras on M locally isomorphic to the sheaf of superfunctions on an open subset of $\mathbb{R}^{n|m}$.

A morphism of two supermanifolds $\varphi : \mathcal{M} \to \mathcal{N}$ is a pair $\varphi = (\tilde{\varphi}, \varphi^*)$, where $\tilde{\varphi} : M \to N$ is a continuous map and a morphism of sheaves

$$\varphi^* : \mathcal{O}_{\mathcal{N}} \to \varphi_* \mathcal{O}_{\mathcal{M}},$$

here $\varphi_* \mathcal{O}_{\mathcal{M}}$ is the induced sheaf on N:
$\varphi_* \mathcal{O}_{\mathcal{M}}(U) = \mathcal{O}_{\mathcal{M}}(\varphi^{-1}(U))$, $U \subset N$.

Consider $\mathcal{M}$ and define the sheaf C^∞_M:

$$C^\infty_M(U) = \mathcal{O}_{\mathcal{M}}(U)/(\mathcal{O}_{\mathcal{M}}(U)_{\bar{1}}).$$

Then C^∞_M defines the structure of a smooth manifold on M.

The inclusion

$$i = (\tilde{i}, i^*) : M \to \mathcal{M}, \quad \tilde{i}(x) = x, \quad i^*(f) = \tilde{f},$$

where

$$f \in \mathcal{O}_{\mathcal{M}}(U) \mapsto \tilde{f} \in C^\infty_M(U) = \mathcal{O}_{\mathcal{M}}(U)/(\mathcal{O}_{\mathcal{M}}(U)_{\bar{1}}).$$

If there exists a splitting $\mathcal{O}_{\mathcal{M}}(U) = C^\infty_M(U) \oplus (\mathcal{O}_{\mathcal{M}}(U)_{\bar{1}})$, then there is an inclusion $C^\infty_M(U) \subset \mathcal{O}_{\mathcal{M}}(U)$, and one considers the projection

$$p = (\tilde{p}, p^*) : \mathcal{M} \to M, \quad \tilde{p}(x) = x, \quad p^*(f) = f,$$

.

Example. Let $E \to M$ be a vector bundle over M, define

$$\mathcal{O}_{\mathcal{M}}(U) = \Gamma(U, \Lambda E), \quad U \subset M.$$

Definition of a supermanifold using local charts

A coordinate chart on a topological space M is a pair $(\mathcal{U}, c)$, where $\mathcal{U} \subset \mathbb{R}^{n|m}$ is a superdomain, and $c : U \to M$ is a homeomorphism on $c(U)$.

Two charts $(\mathcal{U}_1, c_1)$ and $(\mathcal{U}_2, c_2)$ are compatible, if there exists a diffeomorphism

$$\gamma_{12} : (U_{12}, C^\infty\mathcal{U}_1|_{U_{12}}) \to (U_{21}, C^\infty\mathcal{U}_2|_{U_{21}}), \quad \tilde{\gamma}_{12} = c_2^{-1} \circ c_1|_{U_{12}}$$

$$U_{12} = c_1^{-1}(c_1(U_1) \cap c_2(U_2)), \quad U_{21} = c_2^{-1}(c_1(U_1) \cap c_2(U_2))$$

An atlas on a topological space M is a set of compatible charts $((\mathcal{U}_\alpha, c_\alpha), \gamma_{\alpha\beta})$ such that $\cup_\alpha c_\alpha(U_\alpha) = M$, $\gamma_{\beta\alpha} = \gamma_{\alpha\beta}^{-1}$, $\gamma_{\alpha\beta}\gamma_{\beta\delta}\gamma_{\delta\alpha} = \mathrm{id}$.

A supermanifold $\mathcal{M}$ is a pair: a topological space M and an atlas $((\mathcal{U}_\alpha, c_\alpha), \gamma_{\alpha\beta})$.

Product of supermanifolds

If $\mathcal{U}$ and $\mathcal{V}$ are superdomains with the coordinates $x^1, \dots, x^n, \xi^1, \dots \xi^m, \quad y^1, \dots, y^r, \theta^1, \dots, \theta^s$, then $\mathcal{U} \times \mathcal{V}$ is a superdomain with the base $U \times V$ and coordinates $x^1, \dots, x^m, y^1, \dots, y^r, \xi^1, \dots, \xi^m, \theta^1, \dots, \theta^s$.

If $\mathcal{M} = (M, (\mathcal{U}_\alpha, c_\alpha), \gamma_{\alpha\beta})$ and $\mathcal{N} = (N, (\mathcal{V}_\mu, c_\mu), \gamma_{\mu\nu})$ are supermanifolds, then the product $\mathcal{M} \times \mathcal{N}$ is defined by

$$(M \times N, (\mathcal{U}_\alpha \times \mathcal{V}_\mu, c_\alpha \times c_\mu), \gamma_{\alpha\beta} \times \gamma_{\mu\nu}).$$

Theorem of Batchelor (1979).

Let $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a supermanifold. Then there exists a vector bundle $E \to M$ such that $\mathcal{M} \simeq (M, \Gamma(\cdot, \Lambda E))$.

Moreover, there is the following one-to-one correspondence:

$$\left\{ \begin{array}{l} \text{Supermanifolds} \\ \text{of dim. } n|m \text{ mod.} \\ \text{isomorphisms of supermf.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{vector bundles of rank } m \\ \text{over } n\text{-dim. smooth} \\ \text{manifolds mod.} \\ \text{isom. of vector bundles.} \end{array} \right\}$$

Morphisms of supermanifolds are in general not induced by morphisms of vector bundles!

The tangent sheaf: $\mathcal{T}_{\mathcal{M}} = (\mathcal{T}_{\mathcal{M}})_{\bar{0}} \oplus (\mathcal{T}_{\mathcal{M}})_{\bar{1}}$,

$$(\mathcal{T}_{\mathcal{M}})_{\bar{i}}(U) =$$

$$\left\{ X : \mathcal{O}_{\mathcal{M}}(U) \to \mathcal{O}_{\mathcal{M}}(U) \;\middle|\; \begin{array}{c} |X| = \bar{i}, \ X \text{ is } \mathbb{R}\text{-linear} \\ X(fg) = X(f)g + (-1)^{|f||X|} f X(g) \end{array} \right\}$$

The vector fields $\partial_i = \partial_{x^i}$, $\partial_\alpha = \partial_{\xi^\alpha}$ form a local basis of $\mathcal{T}_{\mathcal{M}}(U)$

$\Rightarrow$ $\mathcal{T}_{\mathcal{M}}$ is a locally free sheaf of supermodules over $\mathcal{O}_{\mathcal{M}}$

$x \in M$, the tangent space:

$$T_x\mathcal{M} = \{X : \mathcal{O}_{\mathcal{M},x} \to \mathbb{R} | X(fg) = X(f)g(x) + (-1)^{|f||X|} f(x)X(g)\}.$$

The vectors $(\partial_{x^1})_x, ..., (\partial_{\xi^m})_x$ span $T_x\mathcal{M}$ $((\partial_{x^a})_x f = (\partial_{x^a} f)(x))$.

Note: $(T_x\mathcal{M})_{\bar{0}} = T_x M$.

A Lie supergroup is a supermanifold $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ together with

three morphisms $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad i : \mathcal{G} \to \mathcal{G}, \quad e : \mathbb{R}^{0|0} \to \mathcal{G}$

$$\begin{tikzcd}
 & \mathcal{G} \times \mathcal{G} \arrow[dr, "\mu"] & \\
\mathcal{G} \times \mathcal{G} \times \mathcal{G} \arrow[ur, "\mathrm{id} \times \mu"] \arrow[dr, "\mu \times \mathrm{id}"'] & & \mathcal{G} \\
 & \mathcal{G} \times \mathcal{G} \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{G} \times \mathcal{G} \arrow[dr, "\mu"] & \\
\mathcal{G} \times \mathbb{R}^{0|0} = \mathcal{G} \arrow[ur, "(\mathrm{id}, e)"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "(e, \mathrm{id})"'] & & \mathcal{G} \\
 & \mathcal{G} \times \mathcal{G} \arrow[ur, "\mu"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{G} \times \mathcal{G} \arrow[dr, "\mu"] & \\
\mathcal{G} \arrow[ur, "i \times \mathrm{id}"] \arrow[r] \arrow[dr, "\mathrm{id} \times i"'] & e \arrow[r] & \mathcal{G} \\
 & \mathcal{G} \times \mathcal{G} \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Action of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$: is a morphism

$$a : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$$

such that

$$\begin{tikzcd}
 & \mathcal{G} \times \mathcal{M} \arrow[dr, "a"] & \\
\mathcal{G} \times \mathcal{G} \times \mathcal{M} \arrow[ur, "\mathrm{id}_{\mathcal{G}} \times a"] \arrow[dr, "\mu \times \mathrm{id}_{\mathcal{M}}"'] & & \mathcal{G} \\
 & \mathcal{G} \times \mathcal{M} \arrow[ur, "a"'] &
\end{tikzcd}$$

The Lie superalgebra of a Lie supergroup.

A vectorfield $X \in \mathcal{T}_{\mathcal{G}}(G)$ is called left-invariant if

$$(1 \otimes X) \circ \mu^* = \mu^* \circ X : \mathcal{O}_{\mathcal{G}}(G) \to \mathcal{O}_{\mathcal{G}\times\mathcal{G}}(G \times G).$$

The Lie superalgebra $\mathfrak{g}$ of the Lie supergroup $\mathcal{G}$ is the Lie superalgebra of left-invariant vector fields on $\mathcal{G}$.

Proposition. The vector superspace $\mathfrak{g}$ can be identified with the tangent space $T_e\mathcal{G}$.

The isomorphism is given by

$$X_e \in T_e\mathcal{G} \mapsto X = (1 \otimes X_e) \circ \mu^* \in \mathfrak{g}.$$

Note: $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G.

Super Harish-Chandra pairs.

The Lie supergroup $\mathcal{G}$ defines canonically the pair $(G, \mathfrak{g})$,

$\mathfrak{g}_{\bar{0}} = \mathit{Lie}(G)$;

there exists $\mathrm{Ad} : G \to \mathfrak{gl}(\mathfrak{g})$,

$$\mathrm{Ad}|_{G\times\mathfrak{g}_{\bar{0}}} = \mathrm{Ad}_G, \quad d\mathrm{Ad}|_{\mathfrak{g}_{\bar{0}}\times\mathfrak{g}_{\bar{1}}} = [\cdot,\cdot]_{\mathfrak{g}_{\bar{0}}\times\mathfrak{g}_{\bar{1}}}.$$

Conversely, any such pair $(G, \mathfrak{g})$ defines a Lie supergroup $\mathcal{G}$.

Example. An action $a : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ can be given by an action of G on $\mathcal{M}$ and by a morphism

$$\mathfrak{g} \to (\mathcal{T}_{\mathcal{M}}(M))^0$$

such that the differential of the action of G coincides with the representation of $\mathfrak{g}_{\bar{0}}$.

Example. A representation of $\mathcal{G}$ on a vector superspace V consists of a representation of G on V and of a morphism

$$\mathfrak{g} \to \mathfrak{gl}(V)$$

such that the differential of the representation of G coincides with the representation of $\mathfrak{g}_{\bar{0}}$.

Example.

$$\mathcal{GL}(n|m,\mathbb{R}) = (\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(m,\mathbb{R}), \mathfrak{gl}(n|m,\mathbb{R})),$$

$$\mathrm{OSp}(n|2m,\mathbb{R}) = (\mathrm{O}(n) \times \mathrm{Sp}(2m,\mathbb{R}), \mathfrak{osp}(n|2m,\mathbb{R})).$$

Functor of points.

Let $\mathcal{M}$ be a fixed supermanifold, and $\mathcal{S}$ is another supermanifold.

An $\mathcal{S}$-point of $\mathcal{M}$ is a morphism $\mathcal{S} \to \mathcal{M}$.

The set of $\mathcal{S}$-points of $\mathcal{M}$:

$$\mathcal{M}(\mathcal{S}) = \mathrm{Hom}(\mathcal{S}, \mathcal{M}).$$

Any morphism $\psi : \mathcal{S}_1 \to \mathcal{S}_2$ defines the morphism

$$\mathcal{M}(\psi) : \mathcal{M}(\mathcal{S}_2) \to \mathcal{M}(\mathcal{S}_1), \quad \varphi \mapsto \psi \circ \varphi.$$

The map $\mathcal{S} \mapsto \mathcal{M}(\mathcal{S})$ is a contravariant functor from the category of supermanifolds to the category of sets.

A morphism of supermanifolds $\varphi : \mathcal{M} \to \mathcal{N}$ induces the map

$$\varphi_{\mathcal{S}} : \mathcal{M}(\mathcal{S}) \to \mathcal{N}(\mathcal{S}), \quad \psi \mapsto \varphi \circ \psi.$$

Yoneda's Lemma. For given maps $\{f_{\mathcal{S}} : \mathcal{M}(\mathcal{S}) \to \mathcal{N}(\mathcal{S})\}_{\mathcal{S}}$ that are functorial in $\mathcal{S}$, there exists a unique morphism $\varphi : \mathcal{M} \to \mathcal{N}$ such that $\varphi_{\mathcal{S}} = f_{\mathcal{S}}$.

$\alpha : \mathcal{T} \to \mathcal{S}$

$$\begin{tikzcd} \mathcal{M}(\mathcal{S}) \arrow[r, "f_{\mathcal{S}}"] \arrow[d, "\mathcal{M}(\alpha)"'] & \mathcal{N}(\mathcal{S}) \arrow[d, "\mathcal{N}(\alpha)"] \\ \mathcal{M}(\mathcal{T}) \arrow[r, "f_{\mathcal{T}}"] & \mathcal{N}(\mathcal{T}) \end{tikzcd}$$

Proof. Definition of $\varphi : \mathcal{M} \to \mathcal{N}$:

$\varphi = f_{\mathcal{M}}(\mathrm{id}_{\mathcal{M}})$, where $f_{\mathcal{M}} : \mathcal{M}(\mathcal{M}) \to \mathcal{N}(\mathcal{M})$

Proof of the equality $f_{\mathcal{S}} = \varphi_{\mathcal{S}} : \mathcal{M}(\mathcal{S}) \to \mathcal{N}(\mathcal{S})$: Let $\alpha \in \mathcal{M}(\mathcal{S})$, i.e. $\alpha : \mathcal{S} \to \mathcal{M}$,

$$\begin{tikzcd}
\mathcal{M}(\mathcal{M}) \arrow[r, "f_{\mathcal{M}}"] \arrow[d, "\mathcal{M}(\alpha)"'] & \mathcal{N}(\mathcal{M}) \arrow[d, "\mathcal{N}(\alpha)"] \\
\mathcal{M}(\mathcal{S}) \arrow[r, "f_{\mathcal{S}}"] & \mathcal{N}(\mathcal{S})
\end{tikzcd}$$

$$\varphi_{\mathcal{S}}(\alpha) = \varphi \circ \alpha = f_{\mathcal{M}}(\mathrm{id}_{\mathcal{M}}) \circ \alpha = \mathcal{N}(\alpha)(f_{\mathcal{M}}(\mathrm{id}_{\mathcal{M}})) =$$

$$f_{\mathcal{S}} \circ \mathcal{M}(\alpha)(\mathrm{id}_{M}) = f_{\mathcal{S}} \circ \alpha = f_{\mathcal{S}}(\alpha)$$

Proposition. If $\mathcal{M}$ and $\mathcal{N}$ are supermanifolds, then

$$\mathrm{Hom}(\mathcal{M},\mathcal{N}) = \mathrm{Hom}(\mathcal{O}_{\mathcal{N}}(N),\mathcal{O}_{\mathcal{M}}(M)).$$

Example. The supermanifold $\mathcal{M} = \mathbb{R}^{0|1}$. Any $\mathcal{S}$-point $\varphi : \mathcal{S} \to \mathbb{R}^{0|1}$ is defined by the morphism

$$\varphi^* : C^\infty(\mathbb{R}^{0|1}) = \mathbb{R}\xi \to \mathcal{O}_\mathcal{S}(S),$$

which is given by the odd superfunction $\varphi^*(\xi)$ of $\mathcal{O}_\mathcal{S}(S)_{\bar{1}}$. This superfunction describes elements of $\mathbb{R}^{0|1}(\mathcal{S})$, i.e. it plays the role of usual coordinate on this space, we denote it simply by ξ.

If $\alpha : \mathcal{T} \to \mathcal{S}$ is a morphism than

$$\mathcal{M}(\alpha) : \mathcal{M}(\mathcal{S}) = \mathcal{O}_\mathcal{S}(S)_{\bar{1}} \to \mathcal{M}(\mathcal{T}) = \mathcal{O}_\mathcal{T}(T)_{\bar{1}}, \ \mathcal{M}(\alpha)(\varphi) = \varphi \circ \alpha = \alpha^*,$$

i.e. the map $\mathcal{M}(\alpha)$ is given by $\quad \xi \to \alpha^*(\xi)$.

Thus, $\mathbb{R}^{0|1}(\mathcal{S}) = \mathcal{O}_\mathcal{S}(S)_{\bar{1}}$, $\mathcal{M}(\alpha) = \alpha^*$.

Example. The supermanifold $\mathbb{R}^{n|m}$. Any $\mathcal{S}$-point $\varphi : \mathcal{S} \to \mathbb{R}^{n|m}$ is defined by the morphism

$$\varphi^* : C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Lambda(m) \to \mathcal{O}_{\mathcal{S}}(S),$$

which is given by n even and m odd elements of $\mathcal{O}_{\mathcal{S}}(S)$,

$$\varphi^*(x^1), \dots, \varphi^*(x^n), \varphi^*(\xi^1), \dots, \varphi^*(\xi^m), \quad \text{hence,}$$

$$\mathbb{R}^{n|m}(\mathcal{S}) = \mathcal{O}_{\mathcal{S}}(S)_{\bar{0}}^n \oplus \mathcal{O}_{\mathcal{S}}(S)_{\bar{1}}^m = (\mathcal{O}_{\mathcal{S}}(S) \otimes \mathbb{R}^{n|m})_{\bar{0}}.$$

Let us denote the above functions again by

$$x^1, \dots, x^n, \xi^1, \dots, \xi^m.$$

These coordinates describe the elements of $\mathbb{R}^{n|m}(\mathcal{S})$.

If $\alpha : \mathcal{T} \to \mathcal{S}$ is a morphism then

$$\mathcal{M}(\alpha) : \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{T}), \quad \mathcal{M}(\alpha)\varphi = \varphi \circ \alpha,$$

and $\mathcal{M}(\alpha)$ is defined by $\alpha^*(\varphi^* x^1), ..., \alpha^*(\varphi^* \xi^m)$, i.e. $\mathcal{M}(\alpha) = \alpha^*$.

Any morphism $\varphi : \mathbb{R}^{n|m} \to \mathbb{R}^{r|s}$ is defined by the morphisms $\varphi_{\mathcal{S}} : \mathbb{R}^{n|m}(\mathcal{S}) \to \mathbb{R}^{r|s}(\mathcal{S})$ that can be described in coordinates:

$$\varphi_{\mathcal{S}}(x^1, ..., \xi^m) = (y^1, ..., \theta^s).$$

This gives an explanation to the **symbolic way of calculation**: if $\mathcal{M}$ and $\mathcal{N}$ are supermanifolds with local coordinates $(x, \xi) = (x^i, \xi^\alpha)$ and $(y, \theta) = (y^k, \theta^\beta)$, a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ can be written symbolically

$$\varphi : (x, \xi) \mapsto (y, \theta), \quad y = y(x, \xi), \, \theta = \theta(x, \xi),$$

where in fact $y^k = \varphi^*(y^k) = y^k(x^i, \xi^\alpha)$, $\theta^\beta = \varphi^*(\theta^\beta) = \theta^\beta(x^i, \xi^\alpha)$.

Example. (The supertranslation group of dimension $1|1$)

Consider the supermanifold $\mathbb{R}^{1|1}$ and define the structure of the Lie supergroup on it

$$\mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}, \quad \mu^*(x) = x' + x'' + \xi'\xi'', \quad \mu^*(\xi) = \xi' + \xi''.$$

If we consider (x, ξ) as abstract coordinates on the set of ($\mathcal{S}$-points) of $\mathbb{R}^{1|1}$, then the multiplication is given by

$$((x', \xi'), (x'', \xi'')) \mapsto (x' + x'' + \xi'\xi'', \xi' + \xi'')$$

Exercise. The Lie superalgebra of $\mathbb{R}^{1|1}$ is spanned by the vector fields ∂_x and $D = -\xi\partial_x + \partial_\xi$;

$$[D, D] = 2D^2 = -2\partial_t.$$

A Lie supergroup can be defined in terms of its $\mathcal{S}$-points:

A supermanifold $\mathcal{G}$ is a Lie supergroup iff

for every supermanifold $\mathcal{S}$, $\mathcal{G}(\mathcal{S})$ is a group, and for any morphism $\alpha : \mathcal{T} \to \mathcal{S}$ of supermanifolds, $\mathcal{G}(\alpha) : \mathcal{G}(\mathcal{S}) \to \mathcal{G}(\mathcal{T})$ is a group homomorphism.

The action of $\mathcal{G}$ on $\mathcal{M}$ can be described as the action of the group $\mathcal{G}(\mathcal{S})$ on the set $\mathcal{M}(S)$,

$$a_{\mathcal{S}} : \mathcal{G}(\mathcal{S}) \times \mathcal{M}(S) \to \mathcal{M}(S).$$

Example.

Recall that $\mathrm{Mat}(n|m,\mathbb{R}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$\mathrm{Mat}(n|m,\mathbb{R})_{\bar{0}} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathrm{Mat}(n|m,\mathbb{R})_{\bar{1}} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

We may identify this space with $\mathbb{R}^{n^2+m^2|2nm}$.

We have the following coordinates: x_{ij}, $y_{\alpha\beta}$, $\theta_{i\alpha}$, $\bar{\theta}_{\alpha i}$,

$$x_{ij}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = A_{ij}, \quad \theta_{i\alpha}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = B_{i\alpha}, \dots$$

These coordinates is a basis of $\mathrm{Mat}(n|m,\mathbb{R})^*$.

As the supermanifold,

$$\mathrm{Mat}(n|m,\mathbb{R}) = (\mathrm{Mat}(n,\mathbb{R}) \times \mathrm{Mat}(m,\mathbb{R}), C^\infty(\mathrm{Mat}(n|m,\mathbb{R}))).$$

Define the map

$$\mu = (\tilde{\mu}, \mu^*) : \mathrm{Mat}(n|m,\mathbb{R}) \times \mathrm{Mat}(n|m,\mathbb{R}) \to \mathrm{Mat}(n|m,\mathbb{R}),$$

$\tilde{\mu}$ is the multiplication of matrices, $\mu^* = \mathrm{mult}^*$, where

$\mathrm{mult} : \mathrm{Mat}(n|m,\mathbb{R}) \otimes \mathrm{Mat}(n|m,\mathbb{R}) \to \mathrm{Mat}(n|m,\mathbb{R})$ is the multiplication.

The subset $\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(m,\mathbb{R}) \subset \mathrm{Mat}(n,\mathbb{R}) \times \mathrm{Mat}(m,\mathbb{R})$ is open.

Consider the superdomain

$\mathcal{GL}(n|m,\mathbb{R})$

$$= (\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(m,\mathbb{R}), C^\infty(\mathrm{Mat}(n|m,\mathbb{R}))|_{\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(m,\mathbb{R})}).$$

Together with the multiplication μ it is a Lie supergroup.

Recall that $\mathbb{R}^{n|m}(\mathcal{S}) = (\mathcal{O}_\mathcal{S}(S) \otimes \mathbb{R}^{n|m})_{\bar{0}}$. Hence,

$$\mathrm{Mat}(n|m, \mathbb{R})(\mathcal{S}) = (\mathcal{O}_\mathcal{S}(S) \otimes \mathrm{Mat}(n|m, \mathbb{R}))_{\bar{0}} = \mathrm{Mat}(n|m, \mathcal{O}_\mathcal{S}(S))_{\bar{0}}.$$

The set $\mathrm{Mat}(n|m, \mathcal{O}_\mathcal{S}(S))_{\bar{0}}$ can be viewed as the set of endomorphisms of the $\mathcal{O}_\mathcal{S}(S)_{\bar{0}}$-module $\mathbb{R}^{n|m}(\mathcal{S}) = (\mathcal{O}_\mathcal{S}(S) \otimes \mathbb{R}^{n|m})_{\bar{0}}$.

The subset of automorphisms is the subgroup $\mathrm{GL}(n|m, \mathcal{O}_\mathcal{S}(S))$.

The Lie supergroup $\mathcal{GL}(n|m, \mathbb{R})$ can be described in terms of the functor of point: $\quad S \mapsto \mathcal{GL}(n|m, \mathbb{R})(S) = \mathrm{GL}(n|m, \mathcal{O}_\mathcal{S}(S))$.

The multiplication μ_S is the multiplication of matrices.

The Poincaré supergroup.

Recall that the *Poincaré group*

$$P = \mathrm{O}(1,3) \ltimes \mathbb{R}^{1,3}$$

is the group of isometries of the Minkowski space $\mathbb{R}^{1,3}$; it is the full symmetry of special Relativity.

In quantum field theory, unitary representations of P classify free elementary particles.

Sometimes P is defined as $P = \mathrm{Spin}(1,3) \ltimes \mathbb{R}^{1,3}$.

More generally, $P = \mathrm{Spin}(V) \ltimes V$, $V = \mathbb{R}^{1,n-1}$ or $V = \mathbb{R}^{p,q}$.

The *Poincaré algebra* $\mathfrak{p} = \mathfrak{so}(V) \ltimes V$, $V = \mathbb{R}^{1,n-1}$,

$[A, B] = [A, B]_{\mathfrak{so}(V)}$, $[A, X] = AX$, $[X, Y] = 0$, $A, B \in \mathfrak{so}(V)$, $X, Y \in V$.

N-**extended Poincaré superalgebra** is a Lie superalgebra

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, \quad \mathfrak{g}_{\bar{0}} = \mathfrak{p},$$

$\mathfrak{g}_{\bar{1}}$ is the direct sum of N spinor modules of $\mathfrak{so}(V)$,

$[V, \mathfrak{g}_{\bar{1}}] = 0$, $[\cdot,\cdot]|_{\mathfrak{so}(V)\times\mathfrak{g}_{\bar{1}}}$ is given by the spinor representation,

$[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset V$.

N-**extended Poincaré supergroup** is the Lie supergroup given by the Harish-Chandra pair $(P, \mathfrak{g})$.

In supersymmetric quantum theory, irreducible unitary representations of the Poincaré superalgebra classify elementary superparticles. The restriction of the representation to the underlying Poincaré algebra gives several irreducible representations of it, i.e. a collection of ordinary particles, called *multiplet*. The members of the multiplet are called superpartners of each-other.

Classification of N-extended Poincaré superalgebras:

D.V. Alekseevsky, V. Cortés 1997.

Example. $N = 1$

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, \quad \mathfrak{g}_{\bar{0}} = \mathfrak{p}, \quad \mathfrak{g}_{\bar{1}} = S,$$

it is enough to describe all $\mathfrak{so}(V)$-equivariant maps

$$[\cdot,\cdot]|_{S\otimes S} : \mathrm{Sym}^2 S \to V,$$

the dimension of the space of such maps is the multiplicity of V in the $\mathfrak{so}(V)$-module $\mathrm{Sym}^2 V$.

Let $n = 4, \quad \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{g}_{\bar{1}}, \quad \mathfrak{p} = \mathfrak{so}(1,3) \ltimes \mathbb{R}^{1,3}.$

Minkowski superspace $\mathcal{M}$ is the super Lie group given by super Harish-Chandra pair $(V, V \oplus S)$, where V is the Minkowski space (considered as the abilian Lie group), $V \oplus S \subset \mathfrak{g}$ is the subalgebra, in particular, $[V, V] = [V, S] = 0$, $[S, S] = 0$.

The Poincaré supergroup $\mathcal{P}$ is the group of supersymmetries of $\mathcal{M}$.

The field equations on $\mathcal{M}$ should be invariant w.r.t. the action of $\mathcal{P}$.

$\mathcal{M} = \mathbb{R}^{4|4}$ with the coordinates $x^1, ..., x^4, \xi^1, ..., \xi^4$

$\mathfrak{g} = \mathfrak{p} \oplus S, \quad \mathfrak{p} = \mathfrak{so}(1,3) \ltimes \mathbb{R}^{1,3}, \quad S = \mathbb{R}^4$ (Majorana spinors)

$P_0, ..., P_3 \in \mathbb{R}^{1,3}, \quad Q_1, ..., Q_4 \in S, \quad [Q_\alpha, Q_\beta] = \Gamma^i_{\alpha\beta} P_i$

The representation of the supersymmetry:

$$D_i = \partial_i,$$

$$D_{ij} = x^i \partial_j - x^j \partial_i + \frac{1}{2} (\gamma_{ij})^\alpha_\beta \xi^\beta \partial_\alpha,$$

$$D_\alpha = \frac{1}{2} \Gamma^i_{\alpha\beta} \xi^\beta \partial_i + \partial_\alpha.$$

Super conformal algebra of Wess and Zumino (1974).

This is the first known example of a simple Lie algebra.

$$\mathfrak{g}_{\bar{0}} = \mathfrak{so}(4,2) \oplus \mathfrak{u}(1) \simeq \mathfrak{su}(2,2) \oplus \mathfrak{u}(1), \quad \mathfrak{g}_{\bar{1}} = \mathbb{C}^{2,2},$$

$$\mathfrak{g} = \mathfrak{su}(2,2|1) = \mathfrak{osp}(4,4|2) \cap \mathfrak{sl}(4|2,\mathbb{C})$$

Note that $\mathrm{SO}^0(4,2)$ is the connected group of isometries of AdS^5.

The corresponding homogenious superspace is $\mathrm{AdS}^{5|8}$.